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COHOMOLOGY OF F-GROUPS(1)

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Abstract. Let G be a group of Möbius transformations and V the space of complex polynomials of degree $\leqq$ some fixed even integer. Using the action of G on V defined by Eichler, we compute the dimension of the cohomology space $H^1(G, V)$, first for G an arbitrary F-group (a generalization of Fuchsian group) and then for the free product of finitely many F-groups. These results extend those which Eichler obtained in a 1957 paper, where a correspondence was established between elements of $H^1(G, V)$ and cusp forms on G.

Introduction. In view of recent results in the theory of discontinuous groups, the cohomology of such groups has become an object of study in Riemann surface theory. For a Fuchsian group G with the usual presentation, Eichler [4] computed the dimension of a certain subspace of $H^1(G, V)$, V being a space of complex polynomials, and obtained a correspondence between cohomology classes and cusp forms on G. (The action of G on V is described in §3.) The subspace consists of those cohomology classes represented by cocycles which are trivial on certain generators of G. More recently Bers [1] obtained further results in this direction for Kleinian groups, and it became a matter of interest to know the dimension of the full cohomology space.

In this paper we obtain formulas for the dimension of $H^1(G, V)$ in terms of the parameters occurring in the presentation of G. In §3 and §4, this is carried out for F-groups (these include Fuchsian groups; see §2) and in §5 is extended to the free product of such groups. The computation in §3 is based on a theorem proved in §1, which is essentially a reformulation and slight generalization of a result of Weil [11]. The main results are contained in Theorem 3 and the corollaries to the Lemma of §5.

1. **Cohomology of a finitely presented group.** Let G be a group with generators $a_1, \ldots, a_N, d_1, \ldots, d_m$ and defining relations

$$R = R(a_1, \ldots, d_m) = 1, \qquad d_r^{k_r} = 1, \quad r = 1, \ldots, m.$$

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We may assume the $k_r > 1$. Let G act on a vector space V of finite dimension d over a field F whose characteristic does not divide any of the k_r. (This includes characteristic zero.) That is, we are given a representation ρ of G in Aut (V). Then cohomology spaces $H^k(G, V)$ can be defined for $k \geqq 0$ [2, Chapter X]. However, only $H^1(G, V)$ will be considered in this paper. We recall the definition:

A 1-cocycle of G in V is a mapping f of G into V satisfying $f(ab) = f(a) + \rho(a)f(b)$ for all a, b in G. (In what follows we shall frequently omit the symbol "ρ" in such contexts when there is no danger of confusion.) A 1-coboundary is a mapping g of G into V such that there exists v in V satisfying $g(a) = av - v$ for all a in G. Then $H^1(G, V) = Z^1(G, V)/B^1(G, V)$, where Z^1 and B^1 are the spaces of 1-cocycles and 1-coboundaries, resp.

Let $f \in Z^1(G, V)$. Applying f to the defining relations for G, we get expressions of the form

$$(*) \qquad \begin{aligned} \sum_j A_j f(a_j) + \sum_r B_r f(d_r) &= 0, \\ \sum_{h=1}^{k_r - 1} d_r^h f(d_r) &= 0, \qquad r = 1, \dots, m, \end{aligned}$$

where A_j, B_r are elements of the group ring $Z[G]$ which depend only on R. (V is a $Z[G]$-module in the natural way.) For example, if $R = a_1 a_2 a_1^{-1} a_2^{-1} d_1$, then $0 = f(a_1 a_2 a_1^{-1} a_2^{-1} d_1) = f(a_1) + a_1 f(a_2) + a_1 a_2 f(a_1^{-1}) + a_1 a_2 a_1^{-1} f(a_2^{-1}) + a_1 a_2 a_1^{-1} a_2^{-1} f(d_1)$. But a cocycle always satisfies $f(c^{-1}) = -c^{-1} f(c)$, so the above becomes

$$(1 - a_1 a_2 a_1^{-1}) f(a_1) + (a_1 - a_1 a_2 a_1^{-1} a_2^{-1}) f(a_2) + a_1 a_2 a_1^{-1} a_2^{-1} f(d_1) = 0.$$

Conversely, if f is an arbitrary mapping of $\{a_1, \dots, d_m\}$ into V, then f can be extended to a cocycle if and only if it satisfies (*). This is proved in [11], but a direct proof of a general result of this kind can be given as follows:

Let $(S:N)$ be a presentation of a group G, i.e. S is a set of generators for G and N is a subset of $F(S)$, the free group on S, which generates, as a normal subgroup, the kernel of the natural mapping ϕ of $F(S)$ onto G. (The "natural" mapping is that determined by $s \mapsto s$ for all s in S.) If M is a G-module, we may make M into an $F(S)$-module by setting $wx = \phi(w)x$ for $w \in F(S)$, $x \in M$.

Now an arbitrary mapping f of S into M can be extended uniquely to a cocycle f_1 of $F(S)$ in M in a straightforward way [10, p. 232].

LEMMA. *f is extendible to a cocycle f_2 of G in M if and only if $f_1(r) = 0$ for all r in N. (Note that this is actually a condition on f since $f_1(r)$ can be expressed in terms of the values of f.)*

Proof. The necessity of the condition is obvious. For if f is extendible, it is clear that $f_1(w) = f(\phi(w))$, hence, for $r \in N$, $f_1(r) = f(1) = 0$. To prove sufficiency, suppose f satisfies the given condition. For $g \in G$, define f_2 by $f_2(g) = f_1(w)$, where

w is any element of $F(S)$ such that $\phi(w)=g$. It only remains to show that f_2 is well defined since then it is clearly the required extension of f.

Suppose $\phi(w_1)=\phi(w_2)$, so that $w_1w_2^{-1}\in \mathrm{Kn}\,\phi$, hence is a product of elements of the form ara^{-1} where $r\in N$ and $a\in F(S)$. It suffices to show that $f_1(ara^{-1})=0$, for then $f_1(w_1w_2^{-1})=0$, i.e. $f_1(w_1)-w_1w_2^{-1}f_1(w_2)=0$. But $w_1w_2^{-1}$ acts as identity on V, so $f_1(w_1)=f_1(w_2)$. Now $f_1(ara^{-1})=f_1(a)+af_1(r)-ara^{-1}f_1(a)$. But ara^{-1} acts as identity, and $f_1(r)=0$ by hypothesis, so the right-hand member is 0. This completes the proof of the lemma.

Returning to the original situation, if f is a mapping of $\{a_1,\dots,d_m\}$ into V, then the conditions (*) are precisely the conditions of the form $f_1(r)=0$ which we have just shown to imply the extendibility of f to a cocycle of G in V.

Further, it is well known that $v\in V$ satisfies $\sum_{h=1}^{k_r-1} d_r^h v=0$ (if and) only if $v=(1-d_r)w_r$ for some w_r in V, i.e. if and only if $v\in \mathrm{Im}\,(1-\rho(d_r))=R_r$. For if H is the cyclic group generated by d_r, the condition $\sum d_r^h v=0$ implies by the above lemma that the mapping $d_r\mapsto v$ determines a cocycle of H in V. But $k_rH^1(H,V)=0$ [2, Chapter XII, Proposition 2.5], and so $H^1(H,V)=0$ since the characteristic of F does not divide k_r. Thus the cocycle is a coboundary, i.e. there exists w in V such that $v=(1-d_r)w$.

Summarizing, we have that a mapping f of $\{a_1,\dots,d_m\}$ into V determines a cocycle if and only if it satisfies

$$\sum_{j=1}^{N} A_jf(a_j)+\sum_{r=1}^{m} B_rf(d_r) = 0, \qquad f(d_r)\in R_r = \mathrm{Im}\,(1-\rho(d_r)).$$

Hence the following sequence is exact:

$$0\longrightarrow Z^1(G,V)\xrightarrow{E} V^N\oplus R_1\oplus\cdots\oplus R_m\xrightarrow{D} V\longrightarrow V/\mathrm{Im}\,D\longrightarrow 0,$$

where V^N is the direct sum of N copies of V, and E and D are given by

$$E: f\mapsto (f(a_1),\dots,f(a_N),f(d_1),\dots,f(d_m)),$$
$$D: (u_1,\dots,u_N,v_1,\dots,v_m)\mapsto \sum A_ju_j+\sum B_rv_r.$$

Hence $\dim Z^1(G,V)=(Nd+\sum e_r)-d+i'=(N-1)d+i'+\sum e_r$, where $i'=\dim V/\mathrm{Im}\,D$ and $e_r=\mathrm{rank}\,(1-\rho(d_r))$.

From the definition of D, it is clear that

$$\mathrm{Im}\,D = \sum \mathrm{Im}\,\rho(A_j)+\sum \mathrm{Im}\,[\rho(B_r)\rho(1-d_r)],$$

so that if some $\rho(A_j)$ is invertible, then $i'=0$. This will be the case, for example, if some a_j occurs only once in the relation R. For suppose $R=R_1a_jR_2$, where a_j does not occur in R_1 or R_2. Then $A_j=R_1\in G$ and so $\rho(A_j)$ is invertible.

As for coboundaries, we have the exact sequence $0\to V^G\to V\to B^1(G,V)\to 0$, where $V^G=\{v\in V : av=v \text{ for all } a \text{ in } G\}$, the first mapping is inclusion and the second is given by $v \mapsto f_v$ where $f_v(a) = av - v$ for all a in G. Therefore $\dim B^1 = d - i$, where $i = \dim V^G$. Elements of V^G will be called "fixed points" of V under G. Putting these results together, we have

THEOREM 1. *Let G be a group with generators $a_1, \ldots, a_N, d_1, \ldots, d_m$ and defining relations $R = R(a_1, \ldots, d_m) = 1$, $d_r^{k_r} = 1$, $r = 1, \ldots, m$. Let G act on a vector space V of dimension $d < \infty$ over a field whose characteristic does not divide any of the k_r. Let $f(R) = \sum A_j f(a_j) + \sum B_r f(d_r)$ be the "differentiated" form of R (A_j and B_r are elements of $\mathbf{Z}[G]$ which depend only on R). Then*

(a) $\dim H^1(G, V) = (N-2)d + i + i' + \sum e_r$, *where*

$$i = \dim V^G, \qquad i' = \operatorname{codim}\left(\sum \operatorname{Im} \rho(A_j) + \sum \operatorname{Im}\,[\rho(B_r)\rho(1-d_r)]\right)$$

and $e_r = \operatorname{rank} \rho(1-d_r)$. *($\rho$ is the representation of $\mathbf{Z}[G]$ in* End (V) *corresponding to the action of G on V.)*

(b) *If some a_j occurs only once in R, then $i' = 0$.*

2. **Some properties of F-groups.** In §3, Theorem 1 will be applied to the special case described in the Introduction, viz., G will denote a group of Möbius transformations with generators $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n, d_1, \ldots, d_m$ and defining relations $R = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_n d_1 \cdots d_m = 1$, $d_r^{k_r} = 1$, $r = 1, \ldots, m$. The k_r are assumed > 1. Such groups are called F-groups. They include Fuchsian groups and have been studied in detail in works on discontinuous groups and automorphic functions (see especially [5], [7] and [9]). However, we shall make no nonalgebraic assumptions about G.

In this section, we collect for future reference some properties of these groups which can be determined readily from the presentation. Brief proofs are given for completeness. For this purpose, let $A = 2g - 2 + n + \sum (1 - 1/k_r)$ and call $(g, n, m; k_1, \ldots, k_r)$ the *type* of G. If $A < 0$, G is finite. For the only types satisfying $A < 0$ are $(0, 1, 0;\ )$, $(0, 1, 1; k_1)$, $(0, 0, 1; k_1)$, $(0, 0, 2; k_1, k_2)$, and those types of the form $(0, 0, 3; k_1, k_2, k_3)$ for which $\sum 1/k_r > 1$. It is easy to see that the first four types correspond to finite cyclic groups, and it is known from geometric representations that groups of the last type are finite and not cyclic, [3, p. 68].

On the other hand, if $A \geqq 0$, then G contains an element of infinite order. For if $g > 0$, then a_i and b_i have infinite order since their images in the commutator quotient group have infinite order. If $g = 0$ and $n > 0$, then the relation R may be solved for c_1 and so R and c_1 may be suppressed in the presentation of G. It follows that G is the free product of the cyclic groups generated by $c_2, \ldots, c_n, d_1, \ldots, d_m$. If $n \geqq 2$, then c_2 is an element of infinite order. If $n = 1$, then the conditions, $A \geqq 0$, $g = 0$, imply that $m \geqq 2$. Then $d_1 d_2$ has infinite order.

Finally, suppose $g = n = 0$. Then the condition $A \geqq 0$ implies that $\sum 1/k_r \leqq m - 2$. Now groups of this kind are known to be infinite [3, p. 55]. But by Fenchel's conjecture [6], G has a normal torsion-free subgroup of finite index, so that if G

contained no element of infinite order, it would be finite, a contradiction. This completes the proof that when $A \geqq 0$, G contains an element of infinite order.

Next we claim that when $A \geqq 0$, d_r has order k_r (not merely exponent k_r) for $r=1, \ldots, m$. It suffices to show that G has a quotient in which the image of d_r has order k_r. Suppose first that $g>0$. Then such a quotient is the dihedral group D generated by the Möbius transformations $a: z \mapsto 1/z$ and $b: z \mapsto \zeta/z$, where ζ is a primitive $(2k_r)$th root of unity. Namely, the assignments $a_1 \mapsto a$, $b_1 \mapsto b$, $d_r \mapsto (aba^{-1}b^{-1})^{-1}$, $x \mapsto 1$ for all other generators x, determine a homomorphism of G onto D under which the image of d_r is the transformation $z \mapsto \zeta^2 z$.

Next suppose $n>0$. Then, from the free product representation above, it is clear that d_r has order k_r.

Finally, if $g=n=0$, then the condition $A \geqq 0$ implies that $m \geqq 3$. But Fox [6] shows that if k_1, k_2, k_3 are integers $\geqq 2$, then there exist permutations D_1 and D_2 of orders k_1 and k_2, resp., such that $D_1 D_2$ has order k_3. It is clear then that there is a homomorphism of G onto the group generated by D_1 and D_2 such that the images of d_1, d_2 and d_3 are D_1, D_2 and $(D_1 D_2)^{-1}$, resp.

3. **Cohomology of F-groups.** Let G be an F-group and let q be an integer $\geqq 2$. If $V(q)$ is the space of complex polynomials in one variable of degree $\leqq 2q-2$, we define, as in [4], an action of G on $V=V(q)$ as follows: For $P \in V$ and $\gamma \in G$, $(P\gamma)(z)=P(\gamma(z))/\gamma'(z)^{q-1}$ (γ' is the derivative of γ). Although Theorem 1 was proved for left G-modules for the sake of convenience in consulting references, it is more natural to make G act on the right here, as is done in [1] and [4]. One knows that an analogue to Theorem 1 could be proved to cover such a situation and this will be used in what follows.

In Theorem 2 below, G_1, G_2 and $G_3(q)$ denote the following subgroups of M, the group of all Möbius transformations:

$$\begin{aligned} G_1 &= \{\gamma \in M : \gamma(z) = az \text{ for some } a \neq 0\}, \\ G_2 &= \{\gamma \in M : \gamma(z) = az \text{ or } b/z \text{ for some } a \text{ or } b \neq 0\}, \\ G_3(q) &= \{\gamma \in M : \gamma(z) = \zeta z + b \text{ for some } (q-1)\text{th root of unity } \zeta \text{ and some } b\}. \end{aligned}$$

THEOREM 2. *If G is an F-group of type $(g, n, m; k_1, \ldots, k_m)$ such that $A=2g-2+n+\sum(1-1/k_r) \geqq 0$, then*

(a) *except in the cases described in* (b) *below,*

$$\dim H^1(G, V(q)) = (2g-2+n)(2q-1)+2\sum [q-q/k_r]$$

(*brackets denote the greatest integer function*);

(b) *if G is conjugate* (*in M*) *to a subgroup of G_1 or $G_3(q)$ or if q is odd and G is conjugate to a subgroup of G_2, then*

$$\begin{aligned} \dim H^1(G, V(q)) &= (2g-2)(2q-1)+2+2\sum [q-q/k_r] && \text{if } n = 0, \\ &= (2g-2+n)(2q-1)+1+2\sum [q-q/k_r] && \text{if } n > 0. \end{aligned}$$

REMARK. If we denote the order of a group element x by $o(x)$ and agree that $[q-q/\infty]=q-1$, these formulas take the more convenient forms

(a) $\dim H^1(G, V(q)) = (2g-2)(2q-1)+n+2\sum [q-q/o(f)];$

(b) $$\begin{aligned}\dim H^1(G, V(q)) &= (2g-2)(2q-1)+2+2\sum [q-q/o(f)] \quad \text{if } n = 0,\\ &= (2g-2)(2q-1)+n+1+2\sum [q-q/o(f)] \quad \text{if } n > 0,\end{aligned}$$

where f runs over the c_j and the d_r.

Proof of Theorem 2. In the notation of Theorem 1, $V=V(q)$, $N=2g+n$, $d=2q-1$ and we must compute i, i' and the e_r. We note first that if α is any Möbius transformation, then $H^1(\alpha G\alpha^{-1}, V)\approx H^1(G, V)$, viz., the mapping of $Z^1(G, V)$ onto $Z^1(\alpha G\alpha^{-1}, V)$ given by $f\mapsto f^*$, where $f^*(\alpha\gamma\alpha^{-1})=f(\gamma)\alpha^{-1}$, determines an isomorphism of the cohomology spaces.

Since $A\geqq 0$, there is an element γ_0 in G of infinite order. By transforming G, if necessary, as just indicated, we may assume that γ_0 has one of the following forms:

(a) $\gamma_0(z)=a_0z$, where a_0 is not a root of unity,

(b) $\gamma_0(z)=z+b_0$, $b_0\neq 0$.

We consider these cases separately.

(a) $\gamma_0(z)=a_0z$. Let $P_k(z)=z^k$. Relative to the basis $P_{2q-2},\ldots,P_1,P_0$ of V, $\rho(\gamma_0)$ has a diagonal matrix

$$\operatorname{diag}\{a_0^{q-1}, a_0^{q-2},\ldots,1,a_0^{-1},\ldots,a_0^{-(q-1)}\},$$

so the "fixed point" space of γ_0, i.e., $\{v\in V : v\gamma_0=v\}$, has dimension 1 and is generated by P_{q-1}. Now let γ be any element of G, say $\gamma(z)=(az+b)/(cz+d)$ with $ad-bc=1$. Then $(P_{q-1}\gamma)(z)=(az+b)^{q-1}(cz+d)^{q-1}$. It is now easy to verify, keeping in mind that $ad-bc=1$, that $P_{q-1}\gamma=P_{q-1}$ if and only if *either* (1) $a=d=0$ and q is odd, *or* (2) $b=c=0$. Hence $i=\dim V^G=0$ unless (1) q is odd and G is conjugate to a subgroup of G_2 *or* (2) G is conjugate to a subgroup of G_1. In either of these cases, $i=1$.

As for i', if $n>0$, then the generator c_1 occurs only once in the relation R, so $i'=0$ by part (b) of Theorem 1. The case $n=0$ is treated in [11], where it is shown (for arbitrary V) that $i'=\dim V'^{G'}$. V' is the dual space and G' is the group of transposes $G'=\{{}^t\rho(\gamma) : \gamma\in G\}$. We will write γ^* for ${}^t\rho(\gamma)$.

Let $x'_{2q-2},\ldots,x'_0$ be the dual basis to $P_{2q-2},\ldots,P_0$. The matrix of $\rho(\gamma_0)$ was seen to be diagonal, so that γ_0^* has the same matrix as γ_0, and the fixed point space of γ_0^* is 1-dimensional, generated by x'_{q-1}. Now let $\gamma(z)=(az+b)/(cz+d)$, with $ad-bc=1$, be any element of G, and suppose $x'_{q-1}\gamma^*=x'_{q-1}$. Then, for all j, $\langle P_j, x'_{q-1}\rangle=\langle P_j\gamma, x'_{q-1}\rangle$, where $\langle v, x'\rangle$ is the bilinear form which expresses the duality between V and V'. Setting $j=0$, we get

$$0 = \langle (cz+d)^{2q-2}, x'_{q-1}\rangle = C_{q-1}^{2q-1}(cd)^{q-1},$$

where C_{q-1}^{2q-1} is a binomial coefficient. Hence $cd=0$. Similarly, setting $j=2q-2$, we get that $ab=0$. Thus *either*

(1) $a=d=0$ and q is odd *or*

(2) $b=c=0$.

Conversely, if either of these holds, then $x'_{q-1}\gamma^*=x'_{q-1}$. So $i'=0$ unless $n=0$ and (1) or (2) holds for each $\gamma\in G$. In the latter case, $i'=1$. This completes the discussion of case (a).

(b) $\gamma_0(z)=z+b_0$, $b_0\neq 0$. If $P=P\gamma_0$, then $P(z)=(P\gamma_0)(z)=P(z+b_0)$. Thus P is periodic and hence constant, and so the fixed point space of γ_0 is 1-dimensional, generated by $P_0=1$.

Now let γ be any element of G. Then $(P_0\gamma)(z)=1/\gamma'(z)^{q-1}$, so $P_0\gamma=P_0$ if and only if $\gamma(z)=\zeta z+b$ for some $(q-1)$th root of unity ζ. So $i=0$ unless every γ has this form. Thus, $i=0$ unless G is conjugate to a subgroup of $G_3(q)$, in which case $i=1$.

To find i' (for the case $n=0$; when $n>0$, $i'=0$), note that the matrix of $\rho(\gamma_0)$ relative to $P_{2q-2},\dots,P_0$ has the form

$$\begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

where all elements above the main diagonal, being of the form C_t^k with $0\leqq t\leqq k$, are nonzero. Hence if $v'=\sum \alpha_i x'_i$ is left fixed by γ_0^*, then, applying the transpose of the above matrix to v' and setting the result equal to v', we show in turn that $\alpha_0=0$, $\alpha_1=0,\dots,\alpha'_{2q-3}=0$, so the fixed point space of γ_0^* has dimension 1 and is generated by x'_{2q-2}. If $\gamma(z)=(az+b)/(cz+d)$ with $ad-bc=1$ is any element of G and $x'_{2q-2}\gamma^*=x'_{2q-2}$, then $\langle P_j\gamma, x'_{2q-2}\rangle=\langle P_j, x'_{2q-2}\rangle$ for all j, i.e.,

$$\begin{aligned} a^j c^{2q-2-j} &= 0 \quad \text{if } j\neq 2q-2,\\ &= 1 \quad \text{if } j=2q-2. \end{aligned}$$

Setting $j=0$, we get $c=0$, and setting $j=2q-2$, we get $a^{2q-2}=1$, so that $\gamma\in G_3(q)$. Conversely, if $\gamma\in G_3(q)$, then γ^* has a matrix of the form

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ & * & & 1 \end{bmatrix}$$

so that $x'_{2q-2}\gamma^*=x'_{2q-2}$. Therefore $i'=0$ unless $n=0$ and G is conjugate to a subgroup of $G_3(q)$, in which case $i'=1$. This completes the computation of i and i', viz., they are both zero unless one of the alternatives in the hypothesis of part (b) of Theorem 2 holds. If one of these alternatives holds, then $i=1$ and $i'=1$ or 0 according as $n=0$ or $n>0$.

Finally, to find e_r (see Theorem 1), note first that ρ can be extended to a representation of the full group M of Möbius transformations in an obvious way. Since d_r has order k_r, it is not hard to see that it can be transformed to "diagonal" form, i.e., there exists $\delta_r \in M$, not necessarily in G, such that $(\delta_r d_r \delta_r^{-1})(z)=\zeta_r z$, where ζ_r is a primitive k_rth root of unity. Then $e_r=\text{rank}\,(1-\rho(d_r))=\text{rank}\,(1-\rho(\varepsilon_r))$, where $\varepsilon_r(z)=\zeta_r(z)$. Using the basis $P_{2q-2},\ldots,P_0$, $1-\rho(\varepsilon_r)$ has the diagonal matrix

$$\text{diag}\,\{1-\zeta_r^{q-1}, 1-\zeta_r^{q-2},\ldots,0,1-\zeta_r^{-1},\ldots,1-\zeta_r^{-(q-1)}\},$$

whose rank is $2(q-1-[(q-1)/k_r])$. (Brackets denote the greatest integer function.) It is not hard to verify that the latter expression is equal to $2[q-q/k_r]$. Thus $e_r=2[q-q/k_r]$. A straightforward application of Theorem 1 now completes the proof of Theorem 2.

4. **The exceptional groups.** In order to apply Theorem 2, one must know whether G is one of the exceptional groups covered by part (b) of that theorem. The purpose of this section is to show how this can be determined from the type of G.

Let G be an F-group and suppose $A=2g-2+n+\sum(1-1/k_r)>0$. We shall show that G cannot be a subgroup of G_1, G_2 or $G_3(q)$ for any q. It follows that G cannot be conjugate to such a subgroup since the latter would be an F-group with the same type as G. Since $G_1\subset G_2$ it is sufficient to assume that $G\subset G_2$ or $G_3(q)$ and obtain a contradiction.

If $G\subset G_2$, let $H=\{\gamma\in G:\gamma(z)=az$ for some $a\}$. If $G\subset G_3(q)$, let $H=\{\gamma\in G:\gamma(z)=z+b$ for some $b\}$. In both cases, H is abelian and contains all elements of G of infinite order. Now it is well known (see, e.g., [9, Chapter VII, §3A]) that the condition $A>0$ guarantees that there is a Fuchsian group with the same presentation as G. It follows [8, Theorem 1] that the centralizer of every element (other than 1) is cyclic. Hence H is cyclic, and since G contains elements of infinite order (§2), H is infinite. We have thus proved that the elements of infinite order in G form an infinite cyclic subgroup. Therefore the same is true for any subgroup of G or of a quotient of G. This implies that $g=0$ since otherwise a_1 and b_1 would generate a free abelian subgroup of rank 2 in the commutator quotient group. Further, n must be 0. For if $n>0$, then, as in §2, G is a free product of cyclic groups: $[c_2]*\cdots*[d_m]$. If x and y are generators of any two of these cyclic groups, then xy and yx have infinite order, so they commute. But this can happen only if $x^2=y^2=1$. Thus $n=1$ and $k_r=2$ for all r. Similarly, if x, y, z are generators of any three free factors, then xy and xz commute, which is impossible. Hence $m\leqq 2$. But then $A\leqq 0$, contrary to assumption. Thus $n=0$ as claimed.

We are left with $g=n=0$. We claim that $k_r=2$ for all r. First, if $G\subset G_2$, then all elements in G of finite order are outside H and so have order 2. If $G \subset G_3(q)$, then $d_r(z)=\zeta z+b$ with $\zeta^{q-1}=1$, $\zeta\neq 1$, and if γ is a generator of H, then $\gamma(z)=z+b'$. Then $(d_r\gamma d_r^{-1})(z)=z+\zeta b' \in H$. Therefore $d_r\gamma d_r^{-1}=\gamma^s$ for some integer s, i.e., $z+\zeta b'=z+sb'$. It follows that $\zeta=s=-1$ and so $k_r=2$. Now the conditions, $g=n=0$, $k_1=\cdots=k_m=2$, $A>0$, imply that $m\geqq 5$. But if we add to the presentation of G the relations $d_1d_2d_3=d_4d_5=d_r=1$, $r>5$, we obtain as a quotient of G the free product $K*C$, where K and C have the presentations $\langle d_1, d_2, d_3; d_1^2=d_2^2=d_3^2=d_1d_2d_3=1\rangle$, $\langle d_4, d_5; d_4^2=d_5^2=d_4d_5=1\rangle$, resp. Now K is the Klein 4-group and C is cyclic of order 2. The elements d_1d_4 and d_2d_4 have infinite order in the free product and so, they commute. But it is easy to see this is impossible. This completes the proof that if $A>0$, G is not conjugate to a subgroup of G_2 or $G_3(q)$ for any q, and so formula (a) of Theorem 2 applies.

Next suppose $A=0$. The only types satisfying this are $(1, 0, 0;\ )$, $(0, 2, 0;\ )$, $(0, 1, 2; 2, 2)$, $(0, 0, 3; 2, 4, 4)$, $(0, 0, 3; 2, 3, 6)$, $(0, 0, 3; 3, 3, 3)$ and $(0, 0, 4; 2, 2, 2, 2)$. The first two of these are free abelian groups. As in the proof of Theorem 2, we may assume, possibly after transforming G, that some element of G has the form $\gamma_1(z)=az$, a not a root of unity, or $\gamma_2(z)=z+b$, $b\neq 0$. But the only Möbius transformations which commute with γ_i are those of the same form as γ_i. Hence G is conjugate to a subgroup of G_1 or to a subgroup of $G_3(q)$ for all q. Thus formula (b) of Theorem 2 applies to these two groups.

To dispose of the last five cases listed, we note that if $q\equiv 1 \pmod{l}$, where $l=\text{l.c.m.}\,\{k_1,\ldots,k_m\}$, then $[q-q/k_r]=(q-1)(1-1/k_r)$ and formula (a) of Theorem 2 may be rewritten as $\dim H^1(G, V(q))=A(2q-1)-\sum(1-1/k_r)$. Since $A=0$ and $m>0$, this formula yields a negative dimension for H^1. Thus formula (b) applies for these groups when $q\equiv 1 \pmod{l}$. Conversely, let G be one of these five groups and suppose formula (b) applies, i.e. suppose G is conjugate to a subgroup of $G_3(q)$, or that q is odd and G is conjugate to a subgroup of G_2. (G cannot be conjugate to a subgroup of G_1 since then it would be abelian. But $(0, 1, 2; 2, 2)$ is a free product, hence nonabelian, and the other four groups, being finitely generated by elements of finite order, would be finite if abelian, contradicting the existence of an element of infinite order (§2).) If the former, then all finite orders of elements of G divide $q-1$, so that $q\equiv 1 \pmod{l}$. If G is conjugate to a subgroup of G_2 and q is odd, then G would have to be of type $(0, 1, 2; 2, 2)$ or $(0, 0, 4; 2, 2, 2, 2)$. Namely, in the other three cases, d_2 and d_3 have order >2, hence are in the abelian subgroup G_1. But d_2 and d_3 generate G, so that G would be abelian and therefore, finite, a contradiction. Hence $l=2$ and q, being odd, is $\equiv 1 \pmod{l}$. In these five cases, then, formula (b) applies if and only if $q\equiv 1 \pmod{l}$. If we set $l=1$ when $m=0$, this statement holds for all seven groups for which $A=0$.

Finally, we saw in §2 that when $A<0$, G is finite. If G has order p, then, as noted in the proof of Theorem 1, $pH^1(G, V)=0$, so that $H^1(G, V)=0$ for all q. This completes the proof of the following theorem. The action of G on V is that described at the beginning of §3.

THEOREM 3. *Let G be an F-group of type $(g, n, m; k_1, \ldots, k_m)$, and let $V(q)$ be the space of complex polynomials in one variable of degree $\leqq 2q-2$, q an integer $\geqq 2$. Let $A = 2g-2+n+\sum(1-1/k_r)$ and let $l = \text{l.c.m.}\{k_1, \ldots, k_m\}$. (If $m=0$, we set $l=1$.) Then*

(a) *Except in the cases described in* (b) *and* (c) *below,*

$$\dim H^1(G, V(q)) = (2g-2+n)(2q-1)+2\sum [q-q/k_r] = D.$$

(b) *If $A=0$ and $q \equiv 1 \pmod{l}$,*

$$\begin{aligned} \dim H^1(G, V(q)) &= D+2 \quad \text{if } n = 0, \\ &= D+1 \quad \text{if } n > 0. \end{aligned}$$

(c) *If $A<0$, $H^1(G, V(q))=0$.*

5. **Free products.** Let $G_1, \ldots, G_t$ be groups which act on an abelian group V, i.e. we are given representations ρ_i of the G_i in $\text{Aut}(V)$. If G is the free product of the G_i, then V can be made into a G-module in a natural way. Namely, by the universal mapping property of free products, there exists a unique representation ρ of G in $\text{Aut}(V)$ whose restriction to G_i is ρ_i. If f is a cocycle of G in V, then the restriction $f|G_i$ is clearly a cocycle of G_i in V. We thus have a map $Z^1(G, V) \to \bigoplus_i Z^1(G_i, V)$, viz., $f \mapsto (f|G_1, \ldots, f|G_t)$. Conversely, if $(f_1, \ldots, f_t) \in \bigoplus Z^1(G_i, V)$ and we define f by $f(x_i) = f_i(x_i)$ for x_i in G_i, then the lemma of §1 shows that f is uniquely extendible to a cocycle of G in V. For we may take the (set-theoretic) union of the G_i as a set of generators for G and $\bigcup N_i$ as a set of defining relations, where N_i consists of all relations which hold in G_i. If r_i is a relation in G_i, then, in the notation of the lemma, $f_1(r_i) = f_i(\phi r_i) = 0$, so f is extendible as claimed. It is now clear that the assignment $f \mapsto (f|G_1, \ldots, f|G_t)$ defines an isomorphism of $Z^1(G, V)$ onto $\bigoplus Z^1(G_i, V)$.

As for coboundaries, we have, as in §1, $B^1(G, V) \approx V/V^G$. (It is clear that $V^G = \bigcap V^{G_i}$, but this will not be used.)

LEMMA. *Let G be the free product of groups $G_1, \ldots, G_t$ which act on a vector space V of finite dimension d. Then*

$$\dim H^1(G, V) = \sum \dim H^1(G_i, V) + (t-1)d - \sum \dim V^{G_i} + \dim V^G.$$

(*The action of G on V is that induced by the action of the G_i.*)

Proof. From the preceding discussion, we have

$$\begin{aligned} \dim H^1(G, V) &= \sum \dim Z^1(G_i, V) - \dim V/V^G \\ &= \sum \dim H^1(G_i, V) + \sum \dim V/V^{G_i} - \dim V/V^G, \end{aligned}$$

from which the conclusion follows.

In the following corollaries, notation and terminology are as in §2 and §3. The action of G on V is that described at the beginning of §3. It should also be noted that the property of being trivial or finite cyclic for F-groups is readily determined from the type. Namely, from §2, G is trivial if and only if its type is of the form $(0, 1, 0;\ )$, $(0, 0, 1; k_1)$ or $(0, 0, 2; k_1, k_2)$ with k_1 and k_2 coprime, and G is cyclic of order k if and only if its type is $(0, 1, 1; k)$ or $(0, 0, 2; k_1, k_2)$ with g.c.d. $\{k_1, k_2\} = k$.

COROLLARY 1. *Let G be the free product of $t > 1$ nontrivial F-groups $G_1, \ldots, G_t$, and let q be an integer $\geqq 2$. Then*

(a) *Except in the case described in* (b) *below,*

$$\dim H^1(G, V(q)) = \sum \dim H^1(G_i, V(q)) + (t-1)(2q-1) - \sum \dim V^{G_i}.$$

(b) *If $t=2$ and both groups have order* 2, *then*

$$\begin{aligned} \dim H^1(G, V(q)) &= 1 \quad \textit{if } q \textit{ is even,} \\ &= 0 \quad \textit{if } q \textit{ is odd.} \end{aligned}$$

REMARK. Conditions under which the join of groups of Möbius transformations is the free product are known; see, e.g. [9, Chapter IV, §2].

Proof. Since a nontrivial free product contains an element of infinite order, the argument used in §3 to find V^G applies here. Hence $V^G = 0$ if G is not conjugate to a subgroup of G_2 or $G_3(q)$ (notation as in §3). But the latter groups are metabelian —in fact, an easy computation shows that all commutators are in the abelian subgroup denoted by H in §4—and a free product is never metabelian except for the free product of two groups of order 2. Hence, except for this one case, $V^G = 0$. An application of the lemma now yields part (a).

To prove part (b), note that the free product of two groups or order 2 is itself an F-group, viz., of type $(0, 1, 2; 2, 2)$. Hence Theorem 3 may be used. An application of that theorem yields part (b).

COROLLARY 2. *Given the hypotheses of Corollary* 1, *suppose no G_i has a type of the form* $(0, 0, 3; k_1, k_2, k_3)$ *with $\sum 1/k_r > 1$. Let $(g_i, n_i, m_i; k_{i1}, \ldots, k_{im_i})$ be the type of G_i, let $A_i = 2g_i - 2 + n_i + \sum_r (1 - 1/k_{ir})$ and let $l_i = \text{l.c.m.}\{k_{i1}, \ldots, k_{im_i}\}$ (with the convention that $l_i = 1$ if $m_i = 0$). Then*

$$\dim H^1(G, V(q)) = \sum H_i + (t-1)(2q-1) + \varepsilon,$$

where

$$\begin{aligned} H_i &= 2[q - q/k] - 2q + 1 && \textit{if } G_i \textit{ is cyclic of order } k, \\ &= \dim H^1(G_i, V(q)) - 1 && \textit{if } A_i = 0 \textit{ and } q \equiv 1 \pmod{l_i}, \\ &= \dim H^1(G_i, V(q)) && \textit{otherwise,} \end{aligned}$$

and $\varepsilon = 0$ except when $t=2$, both G_i have order 2 *and q is odd, in which case $\varepsilon = 1$.*

Proof. We may rewrite the conclusion of the lemma as follows:

$$\dim H^1(G, V(q)) = \sum (\dim H^1(G_i, V) - \dim V^{G_i}) + (t-1)(2q-1) + \dim V^G.$$

Then what remains to be shown is that $\dim H^1(G_i, V) - \dim V^{G_i} = H_i$, and that $V^G = 0$ except when q is odd and G is the free product of two groups of order 2, in which case $\dim V^G$ is 1.

Suppose G_i is cyclic of order k, say with generator d_1. Then $V^{G_i} = V^{d_1} = \mathrm{Kn}\,(1-\rho(d_1))$. But rank $(1-\rho(d_1))$ was found in §3 to be $2[q-q/k]$, so $\dim V^{G_i} = 2q-1-2[q-q/k]$. Since $H^1(G_i, V) = 0$ because G_i is finite, this proves that $\dim H^1(G_i, V) - \dim V^{G_i} = H_i$, as required.

This leaves the case $A_i \geqq 0$ since we have ruled out by hypothesis noncyclic groups with $A_i < 0$. In §4 we saw that when $A_i \geqq 0$, $\dim V^{G_i} = 1$ if and only if $A_i = 0$ and $q \equiv 1 \pmod{l_i}$. Otherwise $V^{G_i} = 0$. Combining this remark with an application of Theorem 3, we obtain the required expression for H_i for the case $A_i \geqq 0$.

Finally, we saw above that $V^G = 0$ unless $t = 2$ and both G_i have order 2. In the latter case, the formula of the present corollary yields

$$\begin{aligned} \dim H^1(G, V(q)) &= 1+\varepsilon \quad \text{if } q \text{ is even,} \\ &= -1+\varepsilon \quad \text{if } q \text{ is odd.} \end{aligned}$$

Comparing this with Corollary 1(b), we see that ε must be taken to be 0 if q is even, 1 if q is odd. This completes the proof.

The following corollary applies to the situation which is of greatest interest in the theory of discontinuous groups.

COROLLARY 3. *Let G be the free product of $t > 1$ F-groups $G_1, \ldots, G_t$. Let G_i have type $(g_i, n_i, m_i; k_{i1}, \ldots, k_{im_i})$ and suppose $A_i = 2g_i - 2 + n_i + \sum_r (1-1/k_{ir}) > 0$ for each i. Then*

$$\begin{aligned} \dim H^1(G, V(q)) &= \sum \dim H^1(G_i, V(q)) + (t-1)(2q-1) \\ &= (2q-1) \sum (2g_i - 2 + n_i) + 2 \sum_i \sum_{r_i} [q - q/k_{ir_i}] + (t-1)(2q-1). \end{aligned}$$

Proof. The condition $A_i > 0$ insures that the G_i are infinite, and, as we saw in §4, that $V^{G_i} = 0$. The required conclusion then follows from Corollary 1(a) and Theorem 3.

It is perhaps worth noting that the methods of this section could have been used to find $\dim H^1(G, V(q))$ directly when G is an F-group *with* $n > 0$. A sketch of the argument is as follows. To simplify matters, we assume $A > 0$, although the method may be used for other cases as well.

Since $n > 0$, G is a free product of cyclic groups: $G = [a_1] * \cdots * [b_g] * [c_2] * \cdots * [d_m]$. The first $2g+n-1$ of these are infinite, the last m finite. If C is infinite, $Z^1(C, V) \approx V$ since C is free, so $\dim H^1(C, V) = \dim V^C$. If C is finite, of order k_r, $\dim H^1(C, V) = 0$ and $\dim V^C$, as in the proof of Corollary 2, is $2q-1-2[q-q/k_r]$.

Finally, the condition $A>0$ implies that the number of free factors is $\geqq 2$ and rules out the type $(0, 1, 2; 2, 2)$, so that, as in the proof of Corollary 1, $\dim V^G=0$. Substituting these values in the lemma above, we obtain

$$\dim H^1(G, V(q)) = (2g-2+n)(2q-1)+2\sum_r [q-q/k_r],$$

as we would, of course, have obtained from Theorem 3.

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